Entrance Examination : Ph.D Mathematics, 2012

Hall Ticket Number

Time : 2 hours

Max. Marks. 100

Part A : 25 marks

Part B : 50 marks

Instructions

1. calculators are not allowed.
2. Part A carries 25 marks. Each correct answer carries **1 mark** and each wrong answer carries **-0.33 mark**. If you want to change any answer, cross out the old one and circle the new one. Over written answers will be ignored.
3. Part B carries 50 marks. Instructions for answering Part B are given at the beginning of Part B.
4. Do not detach any pages from this answer book.
5. $\mathbb{R}$ always denotes the set of real numbers, $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of natural numbers and $\mathbb{Q}$ the set of rational numbers.

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Part-A

Answer Part A by circling the correct answer. Each correct answer carries 1 mark and each wrong answer carries -0.33 mark.

1. Let A and B be two $n \times n$ matrices such that $A \neq B$, $A^3 = B^3$ and $A^2B = B^2A$. Then the value of $\det(A^2 + B^2)$ is

   (a) 0. (b) 1. (c) 2. (d) 3/2.

2. The least upper bound of the set $\left\{\frac{(n+1)^2}{2^n} : n \in \mathbb{N}\right\}$ is

   (a) 7/4. (b) 2. (c) 9/4. (d) 5/2.

3. The radius of convergence of the power series $\sum_{n=1}^{\infty} n^{-\sqrt{n}} x^n$ is

   (a) 1. (b) 1/2. (c) 2. (d) ∞.

4. The limit of $(1+x)^{\cot x}$ as $x \to 0$

   (a) exists and its value is 0.
   (b) exists and its value is 1.
   (c) exists and its value is $\exp(1)$
   (d) does not exist.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(f(x)) = -x$ for all $x \in \mathbb{R}$. Then

   (a) f is an injective map.
   (b) f is strictly increasing.
   (c) f is strictly decreasing.
   (d) f is continuous.

6. The function f defined on $\mathbb{R}$ by $f(x) = \begin{cases} 4x & \text{if } x \leq 0, \\ ax^2 + bx + c & \text{if } 0 < x < 1, \\ 3 - 2x & \text{if } x \geq 1. \end{cases}$ is differentiable on $\mathbb{R}$ if

   (a) $a = b = -3$ and $c = 0$.
   (b) $a = -3$, $b = 4$ and $c = 0$.
   (c) $a = 4$, $b = -3$ and $c = 0$.
   (d) $a = b = 4$ and $c = 0$.

7. The sequence (a_n) of reals, where $0 < a_n < 1$ and $a_n(1 - a_{n+1}) > 1/4$ for all $n = 1, 2, \ldots$, converges to .

   (a) 0. (b) 1/2. (c) 1. (d) 2.

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8. The series $\sum_{n=1}^{\infty} \frac{2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}$ converges to

   (a) 1/4. (b) 1/2. (c) 3/4. (d) 1.

9. In which one of these topologies on the real line is it true that the closure of the set $\mathbb{Q}$ of rationals is a countable set properly containing $\mathbb{Q}$

   (a) usual topology generated by $\{(a, b) : a < b \text{ in } \mathbb{R}\}$.

   (b) lower limit topology generated by $\{[a, b) : a < b \text{ in } \mathbb{R}\}$.

   (c) Discrete topology = Power set of $\mathbb{R}$.

   (d) Point mass topology$=\{A \subseteq \mathbb{R} | \sqrt{2} \notin A \text{ or } A \text{ is } \mathbb{R}\}$.

10. If X is a compact metric space, which ones of these follow necessarily?

    (a) X is connected.

    (b) X has a countable base for its topology.

    (c) X is uncountable.

    (d) X is not discrete.

11. Among the four subsets of $\mathbb{R}^2$ given below, only one is connected. Which one is it?

    (a) $\{(x, y) \in \mathbb{R}^2 |$ both x and y are rational$\}$.

    (b) $\{(x, y) \in \mathbb{R}^2 |$ either x or y are rational$\}$.

    (c) $\{(x, y) \in \mathbb{R}^2 | x$ is rational and y is not$\}$.

    (d) $\{(x, y) \in \mathbb{R}^2 |$ neither x nor y is rational$\}$.

12. Which of these four statements is true?

    (a) open interval $(-1, 1)$ and $\mathbb{R}$ are homeomorphic.

    (b) $\mathbb{Q}$ and $\mathbb{Q}^C$ are homeomorphic.

    (c) $\mathbb{Q}$ and $\mathbb{Z}$ are homeomorphic.

    (d) $\mathbb{R}$ and $\mathbb{R} \backslash \{1\}$ are homeomorphic.

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13. Let G be a non abelian group. The order of G could be

(a) 35. (b) 37. (c) 40. (d) 49.

14. The number of idempotent elements in $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$

(a) 2. (b) 4. (c) 6. (d) 8.

15. The set $\{x \in \mathbb{R} : |x+1| = |x| + 1\}$ is same as

(a) $\{x \in \mathbb{R} : x \geq 0\}$. (b) $\{x \in \mathbb{R} : x > 0\}$

(c) whole $\mathbb{R}$. (d) none of these.

16. The critical point $(0,0)$ of the system $\dfrac{dx}{dt} = (y+1)^2 - \cos(x)$, $\dfrac{dy}{dt} = \sin(x+y)$ is

(a) a saddle point. (b) an unstable node.

(c) a stable spiral. (d) none of the above.

17. The number of Jordan canonical forms for 6×6 matrix with minimal polynomial $(x-1)^2(x-2)$ is

(a) 1. (b) 2. (c) 3. (d) none of these.

18. The solution of the Laplace equation in spherical polar coordinates (r, θ, ϕ) is

(a) $\log(r)$. (b) r. (c) $1/r$. (d) r and $1/r$.

19. The equation $u_{xx} + x^2 u_{yy} = 0$ is

(a) elliptic in $\mathbb{R}^2$. (b) elliptic every where except on $x = 0$ axis.

(c) hyperbolic in $\mathbb{R}^2$. (d)hyperbolic every where except on $x = 0$ axis.

20. Let G be a group of order 10. Then

(a) G is an abelian group. (b) G is a cyclic group.

(c) there is a normal proper subgroup. (d) none of these.

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21. The function $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0), \\ 0 & \text{if} \quad (x,y) = (0,0), \end{cases}$ is

(a) continuous at $(0,0)$ but partial derivatives do not exist at $(0,0)$.

(b) continuous at $(0,0)$ and partial derivatives exist at $(0,0)$.

(c) discontinuous at $(0,0)$ but partial derivatives do not exist at $(0,0)$.

(d) none of the above.

22. The number of zero-divisors in the ring $\mathbb{Z}_{24}$ is

(a) 20. (b) 15. (c) 12. (d) none of these.

23. The image of the unit circle under the map $f(z) = 1 + z^2$ is

(a) again the same unit circle.

(b) another circle with a different center but the same radius.

(c) not a circle.

(d) none of the above.

24. Let $A \in M_3(\mathbb{R})$, $A \neq 2I$ and A satisfies the polynomial equation $x^3 - 8 = 0$. Then

(a) minimal polynomial for A is $x^2 + 2x + 4$.

(b) A can be diagonizable.

(c) A is **not** diagonizable.

(d) none of these.

25. The ordinary differential equation $x^2(1-x)^2y'' + xy' + (1-x)^2y = 0$ has

(a) both $x = 0$ and $x = 1$ as regular singular points.

(b) both $x = 0$ and $x = 1$ as irregular singular points.

(c) both $x = 0$ as an ordinary point and $x = 1$ as irregular singular point.

(d) none of the above

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Part B

- Attempt any 10 questions.

1. Suppose $a_n > 0$, $S_n = \sum_{k=1}^{n} a_n$ and $\sum_{n=1}^{\infty} a_n$ diverges.

   (a) Prove that $\frac{a_{n+1}}{S_{n+1}} + \cdots + \frac{a_{n+k}}{S_{n+k}} \geq 1 - \frac{S_n}{S_{n+k}}$ and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$ diverges.

   (b) Prove that $\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$ and deduce that $\sum_{n=1}^{\infty} \frac{a_n}{S_n^2}$ converges.

2. Suppose $f : [0, \infty) \to \mathbb{R}$ is a continuous function and $f(0) = 0$. Let f be differentiable for $x > 0$ and let f' be monotonically increasing. Define $g : (0, \infty) \to \mathbb{R}$ as $g(x) = \frac{f(x)}{x}$. Show that g is monotonically increasing.

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3. Let f be an entire function and suppose that there exists constants $M > 0$, $R > 0$ and $n \in \mathbb{N}$ such that $|f(z)| < M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree less than or equal to n.

4. Show that an entire function has a pole at ∞ of order m if and only if it is a polynomial of degree m.

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5. Consider $\mathbb{R}^3$ with maximum norm, i.e., $||(x, y, x)|| = \max\{|x|, |y|, |z|\}$. Define $T : (\mathbb{R}^3, ||\quad||_\infty) \to (\mathbb{R}^3, ||\quad||_\infty)$ as $T(x, y, z) = (-x + 2y - 3z, 4x - 5y + 6z, -7x + 8y - 9z)$. Find the norm of the operator T.

6. Let $(H, <\quad>)$ be a Hilbert space and (x_n) be a sequence in H. Let $x \in H$ such that $< x_n, y > \to < x, y >$ for all $y \in H$ and $||x_n|| \to ||x||$. Show that $x_n \to x$.

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7. Let (f_n) be a sequence in $L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$, $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. If $f_n \xrightarrow{L^1(\mathbb{R})} f$ and $f_n \xrightarrow{L^2(\mathbb{R})} g$ then show that $f = g$ almost every where.

8. Define a sequence of functions as $f_n(x) = \begin{cases} \exp\left(\frac{1}{n^2x^2-1}\right), & |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Show that $f_n \to 0$ uniformly on $\mathbb{R}\backslash(-1/2, 1/2)$, but f_n does not converge uniformly on $\mathbb{R}$.

9. Determine the Green's function for the boundary value problem $xy'' + y' + f(x) = 0$, $y(1) = 0$ and $\lim_{x \to 0} |y(x)| < \infty$.

10. Determine the stability of the critical point $(0, 0)$ for the system $\frac{dx}{dt} = -y - x\sin^2 x$, $\frac{dy}{dt} = x - y\sin^2 x$.

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11. Let U be a vector space of polynomials of degree less than or equal to three with ordered bases $\{1, x, x^2, x^3\}$ and let V be the vector subspace of the vector space of polynomials of degree less than or equal to 4 with ordered bases $\{x, x^2 + 2x^3 - x^4, x^3 + x^4, 3x^2 + x^3 + 2x^4\}$. Let T be a linear transformation from U to V defined as $Tf = \int_0^x f(t)dt$. Find the matrix of linear transformation T.

12. Let J be a functional of the form $J[y] = \int_{x_0}^{x_1} g(x^2 + y^2)\sqrt{1 + (y')^2}dx$ where g is some function of $x^2 + y^2$. Use the polar coordinate transformation to find the general form of the extremals in terms of g, r and ϕ.

13. Solve $\phi'' + \int_0^x \exp(2x - 2t)\phi'(t)dt = \exp(2x)$, $\phi(0) = 0$ and $\phi'(0) = 1$.

14. Find the inverse Laplace transform of $\frac{1}{s(s+1)}$ using complex inversion formula.

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15. Let $f : pq - z = 0$ and $g : pq - xy = 0$. Show that $z = xy$ is a common solution. Are f and g compatible? Justify.

16. Solve $u_{tt} = u_{xx} + 1, \quad 0 < x < 1, \quad t > 0$ subject to $\begin{cases} u(x,0) = 1, & 0 < x < 1, \\ u_t(x,0) = 1, & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & t > 0. \end{cases}$

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17. Show that $Q_1 = q_1$, $Q_2 = p_2$, $P_1 = p_1 - 2p_2$ and $P_2 = -2q_1 - q_2$ is a canonical transformation.

18. In the real line with usual topology, consider $A = \{x \in \mathbb{R} |$ the integral part of x is even$\}$. Prove that A is neither compact nor connected.

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19. Let $x \in [0,1]$, define $M_x = \{f \in C[0,1]$ such that $f(x) = 0\}$. Show that every maximal ideal of $C[0,1]$ is M_x for some $x \in [0,1]$.

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